EXISTENCE THEORY FOR PERTURBED FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract

In this paper the existence as well as the existence of the extremal solutions for a first order perturbed functional differential inclusions is proved under the mixed generalized Lipschitzity and Carathéodory's conditions.

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1 Statement of Problem

Let $\mathbb{R}$ denote the real line and and let $\mathbb{R}^n$ be an n-dimensional Euclidean space. We define a norm $|\cdot|$ in $\mathbb{R}^n$ by

$$|x| = |x_1| + \cdots + |x_n|$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Let $C = C(I_0, \mathbb{R}^n)$ denote the Banach space of all continuous $\mathbb{R}^n$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_C$ given by

$$\|\phi\|_C = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

For any continuous function x defined on the interval where $J = [-r, a] = I_0 \cup I$ and any $t \in I$ we denote by x_t the element of C defined by

$$x_t(\theta) = x(t + \theta), \quad -r \le \theta \le 0, \;\; 0 \le t \le a.$$

Given a function $\phi \in C$, consider the perturbed functional differential inclusion (in short FDI)

$$\left.\begin{array}{l} x'(t) \in F(t, x_t) + G(t, x_t) \;\;\text{a.e.}\;\; t \in I, \\ x_0 = \phi, \end{array}\right\} \tag{1.1}$$

where $F, G : I \times C \to P_f(\mathbb{R}^n)$ and $P_f(\mathbb{R}^n)$ denotes the class of all nonempty subsets of $\mathbb{R}^n$.

By the solution of FDE (1.1) we mean a function $x \in AC(J, \mathbb{R}^n)$ that satisfies the equations in (1.1), where $AC(J, \mathbb{R}^n)$ is the space of all absolutely continuous functions on J.

The FDE (1.1) is new and the special cases of it have been discussed in the literature since long time. For example, if $F(t, x_t) = \{f(t, x_t)\}$ and $G(t, x_t) = \{g(t, x_t)\}$, then we obtain a functional differential equation

$$\left.\begin{aligned} x'(t) &= f(t, x_t) + g(t, x_t) \quad \text{a.e. } t \in I, \\ x_0 &= \phi, \end{aligned}\right\} \tag{1.2}$$

where $f, g : I \times C \to \mathbb{R}^n$. The more general form of functional differential equation than (1.2) has been discussed in Dhage [7] for existence results. Again when $G \equiv 0$ on $I \times C$, the FDI (1.1) reduces to

$$\left.\begin{aligned} x'(t) &\in F(t, x_t) \quad \text{a.e. } t \in I, \\ x_0 &= \phi, \end{aligned}\right\} \tag{1.3}$$

where $F : I \times C \to P_f(\mathbb{R}^n)$.

The FDI (1.3) has already been discussed in the literature via different methods. The multi-valued version of a fixed point theorem of Krasnoselskii is generally used for proving the existence of solution under the mixed Lipshcitzity and Carathéodory's conditions. See Petrusel [15] and the references therein. In this article we shall prove an existence theorem for FDI (1.1) using a new nonlinear alternative of Schaefer type recently developed in Dhage [6].

2 Auxiliary results

Throughout this paper X will be a Banach space and let $P(X)$ denote the class of all subsets of X. Let $P_f(X)$, $P_{bd,cl}(X)$ and $P_{cp,cv}(X)$ denote respectively the classes of all nonempty, bounded-closed and compact-convex subsets of X. For $x \in X$ and $Y, Z \in P_{bd,cl}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$.

Define a function $H : P_{bd,cl}(X) \times P_{bd,cl}(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The function H is called a Hausdorff metric on X. Note that $\|Y\| = H(Y, \{0\})$.

A correspondence $T : X \to P_f(X)$ is called a multi-valued mapping on X. A point $x_0 \in X$ is called a *fixed point of the multi-valued operator* $T : X \to P_f(X)$ *if* $x_0 \in T(x_0)$. The fixed points set of T will be denoted by $\mathrm{Fix}(T)$.

Definition 2.1 *Let $T : X \to P_{bd,cl}(X)$ be a multi-valued operator. Then T is called a multi-valued contraction if there exists a constant $k \in (0,1)$ such that for each $x, y \in X$ we have*

$$H(T(x), T(y)) \leq k\|x - y\|.$$

The constant k is called a contraction constant of T.

A multi-valued mapping $T : X \to P_f(X)$ is called *lower semi-continuous* (shortly l.s.c.) (resp. *upper semi-continuous* (shortly u.s.c.)) if B is any open subset of X then $\{x \in X \mid Gx \cap B \neq \emptyset\}$(resp.$\{x \in X \mid Gx \subset B\}$) is an open subset of X. The multi-valued operator T is called *compact* if $\overline{T(X)}$ is a compact subset of X. Again T is called *totally bounded* if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. A multi-valued operator $T : X \to P_f(X)$ is called *completely continuous* if it is upper semi-continuous and totally bounded on X, for each bounded $A \in P_f(X)$. Every compact multi-valued operator is totally bounded but the converse may not be true. However the two notions are equivalent on a bounded subset of X.

We apply the following form of the fixed point theorem of Dhage [6] in the sequel.

Theorem 2.1 *Let X be a Banach space, $A : X \to P_{cl,cv,bd}(X)$ and $B : X \to P_{cp,cv}(X)$ two multi-valued operators satisfying*

(a) A is contraction with a contraction constant k, and

(b) B is u.s.c. and completely continuous.

Then either

(i) the operator inclusion $\lambda x \in Ax + Bx$ has a solution for $\lambda = 1$, or

(ii) the set $\mathcal{E} = \{u \in X \mid \lambda u \in Au + Bu,\ \lambda > 1\}$ is unbounded.

We also need the following definitions in the sequel.

Definition 2.2 *A multi-valued map map $F : J \to P_{cp,cv}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.*

Definition 2.3 *A multi-valued map $F : I \times C \to P_{cl}(\mathbb{R}^n)$ is said to be L^1-Carathéodory if*

(i) $t \to F(t,x)$ is measurable for each $x \in C$,

(ii) $x \to F(t,x)$ is upper semi-continuous for almost all $t \in I$, and

(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(I, \mathbb{R}^+)$ such that

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le h_\rho(t), \quad a.e.\ t \in J$$

for all $u \in C$ with $\|u\|_C \le \rho$.

Denote

$$S^1_F(x) = \{v \in L^1(I, \mathbb{R}^n) : v(t) \in F(t, x_t) \ a.e.\ t \in I\}.$$

Then we have the following lemmas due to Lasota and Opial [13].

Lemma 2.1 *If $\dim(X) < \infty$ and $F : J \times X \to P_f(X)$ is L^1-Carathéodory, then $S^1_F(x) \ne \emptyset$ for each $x \in X$.*

Lemma 2.2 *Let X be a Banach space, F an L^1-Carathéodory multi-valued map with $S^1_F \ne \emptyset$ and $\mathcal{K} : L^1(J, X) \to C(J, X)$ be a linear continuous mapping. Then the operator*

$$\mathcal{K} \circ S^1_F : C(J,X) \longrightarrow P_{cp,cv}(C(J,X))$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

3 Existence Result

We consider the following set of assumptions in the sequel.

(H_1) The multi-function $t \to F(t,x)$ is measurable and integrably bounded for each $x \in C$.

(H_2) There exists a function $k \in L^1(I, \mathbb{R}^+)$ such that the multi-function $F : I \times C \to P_{cl,cv,bd}(C(I, \mathbb{R}^n))$ satisfies

$$H(F(t,x), F(t,y)) \le k(t)\|x - y\|_C \quad \text{a.e.} \quad t \in I,$$

for all $x, y \in C$ and $\|k\|_{L^1} < 1$.

(H_3) The multi $G(t,x)$ has compact and convex values for each $(t,x) \in I \times C$.

(H_4) G is L^1-Carathéodory.

(H_5) There exists a function $q \in L^1(I, \mathbb{R})$ with $q(t) > 0$ for a.e. $t \in I$ and a nondecreasing function $\psi : \mathbb{R}^+ \to (0, \infty)$ such that

$$\|G(t,x)\| \le q(t)\psi(\|x\|_C) \ \text{a.e.}\ t \in I,$$

for all $x \in C$.

We use the following lemma in the sequel.

Lemma 3.1 *Suppose that the hypothesis (H_2) holds. Then for any $a \in F(t,x)$,*

$$|a| \leq k(t)\|x\|_C + \|F(t,0)\|, \ t \in I,$$

for all $x \in C$.

Proof: Let $x \in C$ be arbitrary. Then

$$\begin{aligned} \|F(t,x)\| &= H(F(t,x),0) \\ &\leq H(F(t,x),F(t,0)) + H(F(t,0),0) \\ &\leq H(F(t,x),F(t,0)) + \|F(t,0)\|, \end{aligned}$$

for all $t \in I$. Hence for any $a \in F(t,x)$,

$$\begin{aligned} |a| &\leq \|F(t,x)\| \\ &\leq H(F(t,x),F(t,0)) + \|F(t,0)\| \\ &\leq k(t)\|x\|_C + \|F(t,0)\|, \end{aligned}$$

for all $t \in I$. The proof of the lemma is complete. □

Theorem 3.1 *Assume that (H_1)-(H_5) hold. Suppose that*

$$\int_{c_0}^{\infty} \frac{ds}{s+\psi(s)} > \|\gamma\|_{L^1} \tag{3.1}$$

where $c_0 = \displaystyle\int_0^a \|F(s,0)\|\,ds$ and $\gamma(t) = \max\{k(t), q(t)\}$ for $t \in I$. Then the FDI (1.1) has a solution on J.

Proof : The problem of existence of a solution of FDI (1.1) reduces to finding the solution of the integral inclusion

$$\left.\begin{aligned} &x(t) \in \phi(0) + \int_0^t F(s,x_s)\,ds + \int_0^t G(s,x_s)\,ds, \ \ t \in I \\ &x(t) = \phi(t), \quad t \in I_0. \end{aligned}\right. \tag{3.2}$$

We study the integral inclusion (3.2) in the space $X = C(J, \mathbb{R}^n)$ of all continuous $\mathbb{R}^n$-valued functions on J with a supremum norm $\|\cdot\|$. Define two multi-valued maps $A, B : X \to P_f(X)$ by

$$Ax = \begin{cases} \left\{u \in C(I,\mathbb{R}^n) : u(t) = \int_0^t v(s)\,ds, \ \ v \in S_F^1(x)\right\}, & \text{if } t \in I, \\ 0, \quad \text{if } t \in I_0, \end{cases} \tag{3.3}$$

and

$$Bx = \begin{cases} \left\{u \in C(I, \mathbb{R}^n) : u(t) = \phi(0) + \int_0^t v(s)\,ds, \ \ v \in S_G^1(x)\right\}, & \text{if } \ t \in I, \\ \phi(t) \quad \text{if } \ t \in I_0. \end{cases} \tag{3.4}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 2.1 on J.

Step I. First we show that Ax *is closed convex and bounded subset of* X *for each* $x \in X$. This follows easily if we show that the values of Niemytsky operator are closed in $L^1(I, \mathbb{R}^n)$. Let $\{w_n\}$ be a sequence in $L^1(I, \mathbb{R}^n)$ converging to a point w. Then $w_n \to w$ in measure and so, there exists a subset S of positive integers with w_n converging a.e. to w as $n \to \infty$ through S. Now since (H_1) holds, the values of S_F^1 are closed in $L^1(I, \mathbb{R}^n)$. Thus for each $x \in X$ we have that Ax is non-empty and closed subset of X.

We prove that Ax is a convex subset of X for each $x \in X$. Let $u_1, u_2 \in Ax$. Then there exists v_1 and v_2 in $S_F^1(x)$ such that

$$u_j(t) = \int_0^t v_j(s)\,ds, \quad j = 1, 2.$$

Since $F(t, x)$ has convex values, one has for $0 \leq \mu \leq 1$,

$$[\mu v_1 + (1-\mu)v_2](t) \in S_F^1(x)(t), \ \ \forall t \in J.$$

As a result we have

$$[\mu u_1 + (1-\mu)u_2](t) = \int_0^t [\mu v_1(s) + (1-\mu)v_2(t)]\,ds.$$

Therefore $[\mu u_1 + (1-\mu)u_2] \in Ax$ and consequently Ax has convex values in X. From hypothesis (H_1) it follows that Ax is a bounded subset of X for each $x \in X$. Thus we have $A : X \to P_{cl,cv,bd}(X)$.

Step II. Next we show that B *has compact values on* X. Now the operator B is equivalent to the composition $\mathcal{L} \circ S_G^1$ of two operators on $L^1(I, \mathbb{R}^n)$, where $\mathcal{L} : L^1(I, \mathbb{R}^n) \to P(C(I, \mathbb{R}^n))$ is defined by

$$\mathcal{L}v(t) = \phi(0) + \int_0^t v(s)\,ds.$$

To show B has compact values, it is enough to show that the Niemytskii operator has compact values on $L^1(I, \mathbb{R}^n)$. Let $x \in X$ be arbitrary and let $\{v_n\}$ be any sequence in $S_G^1(x)$. Then $v_n(t) \in G(t, x_t)$ a.e. for $t \in I$. Since $G(t, x_t)$ is compact, there exists a subset S of positive integers such that $v_n(t) \to v(t)$ as $n \to \infty$ through S and

$v(t) \in G(t, x_t)$ a.e. for $t \in I$. As a result we have that $v_n \to v$ as $n \to \infty$ through S. Hence $S^1_G(x)$ is compact and consequently by superposition principle Bx is compact for each $x \in X$. See Appell [2] and the references therein. Further as in the case of operator A it can be shown that B has convex values on X. Thus we have $B : X \to P_{cp,cv}(X)$.

Step III. We show that *A is a multi-valued contraction on X*. Let $x, y \in X$ and $u_1 \in Ax$. Then $u_1 \in C(I, \mathbb{R}^n)$ and $u_1(t) = \int_0^t v_1(s)\,ds$ for some $v_1 \in S^1_F(x)$. Since $H(F(t, x_t), F(t, y_t) \leq k(t)\|x_t - y_t\|_C$, one obtains that there exists $w \in F(t, y_t)$ such that $\|v_1(t) - w\| \leq k(t)\|x_t - y_t\|_C$. Thus the multi-valued operator U defined by $U(t) = S^1_F(y)(t) \cap K(t)$, where

$$K(t) = \{w \in \mathbb{R}^n |\ \|v_1(t) - w\| \leq k(t)\|x_t - y_t\|_C\}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U (which exists by Kuratowski-Ryll-Nardzewski's selection theorem. See [3]). Then $v_2 \in F(t, y_t)$ and $\|v_1(t) - v_2(t)\| \leq k(t)\|x_t - y_t\|_C$ a.e. on I.

Define $u_2(t) = \displaystyle\int_0^t v_2(s)\,ds$. It follows that $u_2 \in Ay$ and

$$\begin{aligned}
\|u_1(t) - u_2(t)\| &\leq \left\| \int_0^t v_1(s)\,ds - \int_0^t v_2(s)\,ds \right\| \\
&\leq \int_0^t \|v_1(s) - v_2(s)\|\,ds \\
&\leq \int_0^t k(t)\|x_s - y_s\|_C\,ds \\
&\leq \|k\|_{L^1}\|x - y\|.
\end{aligned}$$

Taking the supremum over t, we obtain

$$\|u_1 - u_2\| \leq \|k\|_{L^1}\|x - y\|.$$

From this and the analogous inequality obtained by interchanging the roles of x and y we get that

$$H(A(x), A(y)) \leq \|k\|_{L^1}\|x - y\|,$$

for all $x, y \in X$. This shows that A is a multi-valued contraction, since $\|k\|_{L^1} < 1$.

Step IV. Now we show that *the multi-valued operator B is completely continuous on X*. First we show that B maps bounded sets into bounded sets in X. To see this, let Q be a bounded set in X. Then there exists a real number $r > 0$ such that $\|x\| \leq r, \forall x \in Q$.

Now for each $u \in Bx$, there exists a $v \in S^1_G(x)$ such that

$$u(t) = \phi(0) + \int_0^t v(s)\,ds.$$

Then for each $t \in I$,

$$\begin{aligned} |u(t)| &\leq |\phi(0)| + \int_0^t |v(s)|\, ds \\ &\leq \|\phi\|_C + \int_0^t h_r(s)\, ds \\ &\leq \|\phi\|_C + \|h_r\|_{L^1}. \end{aligned}$$

This further implies that

$$\|u\| \leq \|\phi\|_C + \|h_r\|_{L^1}$$

for all $u \in Bx \subset \bigcup B(Q)$. Hence $\bigcup B(Q)$ is bounded.

Next we show that B maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $u \in Bx$ for some $x \in Q$. Then there exists $v \in S^1_G(x)$ such that

$$u(t) = \phi(0) + \int_0^t v(s)\, ds.$$

Then for any $t_1, t_2 \in I$ with $t_1 \leq t_2$ we have

$$\begin{aligned} |u(t_1) - u(t_2)| &\leq \left| \int_0^{t_1} v(s)\, ds - \int_0^{t_2} v(s)\, ds \right| \\ &= \int_{t_1}^{t_2} |v(s)|\, ds \\ &\leq \int_{t_1}^{t_2} h_r(s) ds. \end{aligned}$$

If $t_1, t_2 \in I_0$ then $|u(t_1) - u(t_2)| = |\phi(t_1) - \phi(t_2)|$. For the case where $t_1 \leq 0 \leq t_2$ we have that

$$\begin{aligned} |u(t_1) - u(t_2)| &\leq \left| \phi(t_1) - \phi(0) - \int_0^{t_2} v(s)\, ds \right| \\ &\leq |\phi(t_1) - \phi(0)| + \int_0^{t_2} |v(s)|\, ds \\ &\leq |\phi(t_1) - \phi(0)| + \int_0^{t_2} h_r(s) ds. \end{aligned}$$

Hence, in all cases, we have

$$|u(t_1) - u(t_2)| \to 0 \text{ as } t_1 \to t_2.$$

As a result $\bigcup B(Q)$ is an equicontinuous set in X. Now an application of Arzelá-Ascoli theorem yields that the multi B is totally bounded on X.

Step V. Next we prove that B *has a closed graph.* Let $\{x_n\} \subset X$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bx_n$ for each $n \in \mathbb{N}$ such that $y_n \longrightarrow y_*$. We will show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = \phi(0) + \int_0^t v_n(s)\,ds.$$

Consider the linear and continuous operator $\mathcal{K} : L^1(X) \to C(X)$ defined by

$$\mathcal{K}v(t) = \int_0^t v_n(s)\,ds.$$

Now

$$\begin{aligned} |y_n(t) - \phi(0) - (y_*(t) - \phi(0))| &\leq |y_n(t) - y_*(t)| \\ &\leq \|y_n - y_*\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

From Lemma 2.2 it follows that $(\mathcal{K} \circ S^1_G)$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$y_n(t) - \phi(0) \in (\mathcal{K} \circ S^1_F(x_n)).$$

As $x_n \longrightarrow x_*$ and $y_n \longrightarrow y_*$, there is a $v \in S^1_G(x_*)$ such that

$$y_*(t) = \phi(0) + \int_0^t v_*(s)\,ds.$$

Hence the multi B is an upper semi-continuous operator on X.

Step VI. Finally we show that the set

$$\mathcal{E} = \{u \in X : \lambda u \in Au + Bu \text{ for some } \lambda > 1\}$$

is bounded.

Let $u \in \mathcal{E}$ be any element. Then there exists $v_1 \in S^1_F(u)$ and $v_2 \in S^1_G(u)$ such that

$$u(t) = \lambda^{-1}\phi(0) + \lambda^{-1}\int_0^t v_1(s)\,ds + \lambda^{-1}\int_0^t v_2(s)\,ds.$$

Then

$$\begin{aligned} |u(t)| &\leq \phi(0) + \int_0^t |v_1(s)|\,ds + \int_0^t |v_2(s)|\,ds \\ &\leq \phi(0) + \int_0^t (k(s)\,\|u_s\|_C + \|F(s,0\|)\,ds + \int_0^t q(s)\psi(\|u_s\|_C)\,ds. \end{aligned}$$

Put $w(t) = \max\{|u(s)| : -r \leq s \leq t\}$, $t \in I$. Then $\|u_t\|_C \leq w(t)$ for all $t \in I$ and there is a point $t^* \in [-r, t]$ such that $w(t) = u(t^*)$. Hence we have

$$w(t) \;=\; |u(t^*)|$$

$$
\begin{aligned}
&\leq \int_0^{t^*} k(s)\|u_s\|_C\, ds + \int_0^{t^*} \|F(s,0)\|\, ds + \int_0^{t^*} q(s)\psi(\|u_s\|_C)\, ds \\
&\leq c_0 + \int_0^t k(s)w(s)\, ds + \int_0^t q(s)\psi(w(s))\, ds \\
&\leq c_0 + \int_0^t \gamma(s)(w(s) + \psi(w(s))\, ds,
\end{aligned}
$$

where $\gamma(t) = \max\{k(t), q(t)\}$ for $t \in I$.

Let

$$m(t) = c_0 + \int_0^t \gamma(s)(w(s) + \psi(w(s))\, ds, \quad t \in I.$$

Then we have $w(t) \leq m(t)$ for all $t \in I$. Differentiating w.r.t. to t, we obtain

$$m'(t) = \gamma(t)(w(t) + \psi(w(t)), \text{ a.e. } t \in I,\ m(0) = c_0.$$

This further implies that

$$m'(t) \leq \gamma(t)(m(t) + \psi(m(t)), \text{ a.e. } t \in I,\ m(0) = c_0,$$

that is,

$$\frac{m'(t)}{m(t) + \psi(m(t))} \leq \gamma(t) \text{ a.e. } t \in J,\ m(0) = c_0.$$

Interesting from 0 to t we get

$$\int_0^t \frac{m'(s)}{m(t) + \psi(m(t))}\, ds \leq \int_0^t \gamma(s) ds.$$

By the change of variable,

$$\int_{c_0}^{m(t)} \frac{ds}{s + \psi(s)} \leq \|\gamma\|_{L^1} < \int_{c_0}^{\infty} \frac{ds}{s + \psi(s)}\, ds.$$

Hence there exists a constant M such that

$$w(t) \leq m(t) \leq M \text{ for all } t \in I.$$

Now from the definition of w it follows that

$$\|u\| = \sup_{t \in [-r,a]} |u(t)| = w(a) \leq m(a) \leq M,$$

for all $u \in \mathcal{E}$. This shows that the set $\mathcal{E}$ is bounded in X. As a result the conclusion (ii) of Theorem 2.1 does not hold. Hence the conclusion (i) holds and consequently (3.2) or equivalently FDI (1.1) has a solution x on J. This completes the proof. □

4 Existence of Extremal Solutions

In this section we shall prove the existence of maximal and minimal solutions of the FDI (1.1) under suitable monotonicity conditions on the multi-functions involved in it. We define the usual co-ordinate-wise order relation " $\leq$ " in $\mathbb{R}^n$ as follows. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ be any two elements. Then by " $x \leq y$ " we mean $x_i \leq y_i$ for all $\forall i$, $i = 1, \cdots, n$. We equip the space $C(J, \mathbb{R}^n)$ with the order relation $\leq$ defined by the cone K in $C(J, \mathbb{R}^n)$, that is,

$$K = \{x \in C(J, \mathbb{R}^n) \mid x(t) \geq 0, \ \ \forall\, t \in J\}. \tag{4.1}$$

It is known that the cone K is normal in $C(J, \mathbb{R}^n)$. The details of cones and their properties may be found in Heikkila and Lakshmikantham [11]. Let $a, b \in C(J, \mathbb{R}^n)$ be such that $a \leq b$. Then by an order interval $[a, b]$ we mean a set of points in $C(J, \mathbb{R}^n)$ given by

$$[a, b] = \{x \in C(J, \mathbb{R}^n) \mid a \leq x \leq b\}. \tag{4.2}$$

Let $D, Q \in P_{cl}(C(J, \mathbb{R}^n))$. Then by $D \leq Q$ we mean $a \leq b$ for all $a \in D$ and $b \in Q$. Thus $a \leq D$ implies that $a \leq b$ for all $b \in Q$ in particular, if $D \leq D$, then it follows that D is a singleton set.

Definition 4.1 *Let X be an ordered Banach space. A mapping $T : X \to P_{cl}(X)$ is called isotone increasing if $x, y \in X$ with $x < y$, then we have that $Tx \leq Ty$.*

We use the following fixed point theorem in the proof of main existence result of this section.

Theorem 4.1 (Dhage [7]). *Let $[a, b]$ be an order interval in a Banach space and let $A, B : [a, b] \to P_{cl}(X)$ be two multi-valued operators satisfying*

(a) *A is multi-valued contraction,*

(b) *B is completely continuous,*

(c) *A and B are isotone increasing, and*

(d) *$Ax + Bx \subset [a, b]$, $\ \forall\, x \in [a, b]$.*

Further if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a least fixed point x_* and a greatest fixed point x^* in $[a, b]$. Moreover $x_* = \lim_n x_n$ and $x^* = \lim_n y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by

$$x_{n+1} \in Ax_n + Bx_n, \ x_0 = a \quad \text{and} \quad y_{n+1} \in Ay_n + By_n, \ y_0 = b.$$

We need the following definitions in the sequel.

Definition 4.2 *A solution $a \in C(J, \mathbb{R}^n)$ is called a lower solution of the FDI (1.1) if $a'(t) \leq v_1(t)+v_2(t)$ for all $t \in I$ and $a(t) \leq \phi(t)$ for all $t \in I_0$, where $v_1, v_2 \in L^1(I, \mathbb{R}^n)$ such that $v_1(t) \in F(t, a_t)$ and $v_2(t) \in G(t, a_t)$ almost everywhere $t \in I$. Similarly an upper solution b of the FDI (1.1) is defined.*

Definition 4.3 *A solution x_M of the FDI (1.1) is said to be maximal if x is any other solution of FDI (1.1) on J, then we have $x(t) \leq x_M(t)$ for all $t \in J$. Similarly a minimal solution of the FDI (1.1) is defined.*

We consider the following assumptions in the sequel.

(H_6) The multi-functions $F(t, x)$ and $G(t, x)$ are nondecreasing in x almost everywhere for $t \in I$.

(H_7) The FDI (1.1) has a lower solution a and an upper solution b with $a \leq b$.

Theorem 4.2 *Assume that the hypotheses (H_1)-(H_7) hold. Then the FDI (1.1) has minimal and maximal solutions on J.*

Proof : Let $X = C(J, \mathbb{R}^n)$ and consider the order interval $[a, b]$ in X which is well defined in view of hypothesis (H_7). Define two operators $A, B : [a, b] \to P_{cl}(X)$ by (3.3) and (3.4) respectively. It can be shown, as in the proof of Theorem 3.1, that A and B define the multi-valued operators $A : [a, b] \to P_{cl,cv,bd}(X)$ and $B : [a, b] \to P_{cp,cv}(X)$. It is also similarly shown that A and B are respectively multi-valued contraction and completely continuous on $[a, b]$. We shall show that A and B are isotone increasing on $[a, b]$. Let $x \in [a, b]$ be such that $x \leq y, x \neq y$. Then by (H_6), we have

$$\begin{aligned} Ax(t) &= \left\{u(t) : u(t) = \int_0^t v(s)\,ds,\ v \in S_F^1(x)\right\} \\ &\leq \left\{u(t) : u(t) = \int_0^t v(s)\,ds,\ v \in S_F^1(y)\right\} \\ &= Ay(t), \end{aligned}$$

for all $t \in I$ and $Ax(t) = 0 = Ay(t)$ for all $t \in I_0$. Hence $Ax \leq Ay$. Similarly by (H_6), we have

$$\begin{aligned} Bx(t) &= \left\{u(t) : u(t) = \phi(0) + \int_0^t v(s)\,ds,\ v \in S_G^1(x)\right\} \\ &\leq \left\{u(t) : u(t) = \phi(0) + \int_0^t v(s)\,ds,\ v \in S_G^1(y)\right\} \\ &= By(t), \end{aligned}$$

for all $t \in I$ and $Bx(t) = \phi(t) = By(t)$ for all $t \in I_0$. Hence $Bx \leq By$. Thus A and B are isotone increasing on $[a, b]$. Finally let $x \in [a, b]$ be any element. Then by (H_7),

$$a \leq Aa + Ba \leq Ax + Bx \leq Ab + Bb \leq b,$$

which shows that $Ax + Bx \in [a, b]$ for all $x \in [a, b]$. Thus the multi-valued operator A and B satisfy all the conditions of Theorem 4.1 to yield that the operator inclusion and consequently the FDI (1.1) has maximal and minimal solutions on J. This completes the proof. $\square$

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